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LETTER TO THE EDITOR

Fractional strings hypothesis and non-simple laced integrable models

Márcio José Martins†‡

Department of Physics, University of California, Santa Barbara, CA 93106, USA

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Abstract. We discuss the ground state and some elementary excitations of the B_p non-simple laced integrable models. The zeros of the associated Bethe ansatz equations are characterized by a special distribution in the complex plane, which is different from the usual string hypothesis. The introduction of this new solution makes it possible to calculate exactly various properties in the thermodynamic limit. The finite-size corrections to the eigenspectrum strongly suggest that the B_p Wess–Zumino–Witten–Novikov models are the underlying conformal field theory for these non-simple laced models.

Among the important outcomes of conformal invariance are the direct relations between the eigenspectra of conformal invariant systems and their associated central charges and conformal dimensions [1]. In principle, one can infer the associated operator content for a given conformal model by using exact diagonalization or Lanczos approaches. However, these traditional procedures are limited by the rapidly increasing size of the associated Hamiltonians or transfer matrices, respectively, and in many cases do not give conclusive results [2–4]. Recently, 2D integrable models solved by the Bethe ansatz approach have provided explicit realizations of conformal field theories. In these gapless models, numerical [11–14, 16]/analytical [5–10] manipulations of the associated Bethe ansatz equations (BAE) can determine the conformal anomaly and the conformal dimensions to within high precision. Most of these calculations have been performed in the simple-laced exactly solved (SLES) models. All the results point to the fact that these SLES models are the lattice realizations of the associated Wess–Zumino–Witten–Novikov (WZWN) conformal field theories. Although the BAE have been established/conjectured for some non-simple laced exactly solved (NSLES) models [17], they have not been explicitly manipulated either in the thermodynamic limit ($L \to \infty$) or to obtain the finite-size corrections to the eigenspectrum. In addition, several authors [15, 18–21] have recently applied the thermodynamic Bethe ansatz method to study the ultraviolet behaviour of perturbed conformal field theories. This technique seems to work only for simple laced models [21]. Motivated by these facts and considering that previous studies have only concentrated on the simple laced models, we believe it important to analyse the BAE in the non-simple laced cases. In this letter we choose the B_p NSLES models (in their fundamental representation) in order to show some exotic properties of the zeros distribution of the associated BAE. We show for the first time that the ground states are characterized

† On leave from: Departamento de Fisica, Universidade Federal de S Carlos, CP 676, S Carlos 13560, Brazil.
‡ Electronic mail address: martins@sarek.physics.ucsb.edu

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by some special configurations; we have denoted this by the fractional string hypothesis (FSH) to differentiate it from the traditional string hypothesis [22]. As a consequence, we calculate the ground state energy, the densities of the BAEs zeros, and the sound velocities. We also discuss the finite-size scaling properties for the ground state and for some lowest excitations.

The BAE and the eigenspectrum (E) of these B_p models are

$$(f_1(\lambda_j^1))^L = -\prod_{k=1}^{m_1} f_2(\lambda_j^1-\lambda_k^1)\prod_{k=1}^{m_2} f_{-1}(\lambda_j^1-\lambda_k^2)$$

$$1 = -\prod_{k=1}^{m_l} f_2(\lambda_j^l-\lambda_k^l)\prod_{\varepsilon=1,-1}\prod_{k=1}^{m_l+\varepsilon} f_{-1}(\lambda_j^l-\lambda_k^{l+\varepsilon}) \qquad l=2,3,\ldots,p-1$$

$$1 = -\prod_{k=1}^{m_p} f_1(\lambda_j^p-\lambda_k^p)\prod_{k=1}^{m_{p-1}} f_{-1}(\lambda_j^p-\lambda_k^{p-1}) \tag{1}$$

$$E = -\frac{1}{2}\sum_{j=1}^{m_1}\frac{1}{(\lambda_j^1)^2+\frac{1}{4}} \qquad f_a(x)=\frac{x-\mathrm{i}a/2}{x+\mathrm{i}a/2} \tag{2}$$

where m_l, $l=1,2,\ldots,p$ are integer numbers that specify the particular sector in the Hilbert space.

After a careful study of (1) and of the corresponding B_p NSLES Hamiltonian [17], we find that the ground state is characterized by a set $\{\lambda_1=\alpha_j^1,\ldots,\lambda_j^{p-1}=\alpha_j^{p-1}, \lambda_j^p=\alpha_j^p\pm\frac{1}{4}\mathrm{i}(1+\delta_j)\}$, where α_j^i, $i=1,2,\ldots,p$ are real parameters. In table 1 we show the complex part of the λ_j^p zeros for $p=2,3$ and $L=40$. Extrapolating these results using VBS approximates [23], we conclude that $\delta_j\to 0$ when $L\to\infty$ and $\lambda_j^p=\alpha_j^p\pm\frac{1}{4}\mathrm{i}$. This picture is different from that of the SLES (in their fundamental representation) case in two ways: first, in the SLES models the ground state is characterized by a set of only real solutions; and second, because $\lambda_j^p=\alpha_j^p\pm\frac{1}{4}\mathrm{i}$ does not fit the traditional string hypothesis. The imaginary part of our solution for λ_j^p is just one half of the known 2-string hypothesis, hence justifying the FSH name.

In order to calculate the ground state energy per particle (e_∞^p), the densities of roots ($\sigma^k(x)$), and the sound velocities (ζ), we substitute back the FSH into (1). After some algebraic manipulations, differentiating with respect to α_j^i, and using the definition $\sigma^k=1/L(\alpha_j^k-\alpha_{j-1}^k)$, we generate p-coupled integral equations for the densities σ^k.

Table 1. The imaginary parts of the λ_j^p's zeros for $p=2,3$ and $L=40$.

j	$p=2$	$p=3$
1	0.297 6708	0.310 1478
2	0.267 6077	0.279 1826
3	0.260 8155	0.267 9395
4	0.257 9259	0.263 1732
5	0.256 3760	0.260 6084
6	0.255 4448	0.259 0641
7	0.254 8551	0.258 0851
8	0.254 4803	0.257 4624
9	0.254 2566	0.257 0905
10	0.254 1517	0.256 9161

These integral equations are solved by elementary Fourier techniques, and we summarize the results here

$$\sigma^k(x)=\frac{[4/(2p-1)]\cos[\pi(2p-1-k)/(2p-1)]\cosh[2x\pi/(2p-1)]}{\cosh[4x\pi/(2p-1)]+\cos[(2p-1-k)\pi/(2p-1)]}$$

$$k=1,2,\ldots,p-1 \tag{3}$$

$$\sigma^p(x)=\frac{1}{(2p-1)\cosh[2x\pi/(2p-1)]}. \tag{4}$$

The ground state energy per particle is then calculated using (1), (2), and the result is

$$e^p_\infty=-\frac{\psi[2p+1/2(2p-1)]-\psi[1/(2p-1)]+2\ln(2)}{2p-1} \tag{5}$$

where $\psi(x)$ is the Euler psi-function. As for any problem which is solved by Bethe's ansatz, the low-lying excited states are obtained introducing holes into the ground state picture. This generates p excitation branches, and all of them have a common slope when the total momentum (K) goes to zero. The low-momentum dispersion relation ($\varepsilon(K)$) is $\varepsilon(K)=[\pi/(2p-1)]|K|$, $K\sim 0$. The parameter $\zeta=\pi/(2p-1)$ is precisely the sound velocity for these B_p models.

The stability of these solutions depends explicitly upon the FSH for the λ^p_j variables when $L\to\infty$. To check this stability, we modify the FSH by $\lambda^p_j=\alpha^p_j\pm\frac{1}{4}\mathrm{i}(1+\delta_j)$ in order to take into account effects of finite-size corrections. The δ_j deviations can be computed adopting the technique developed in [24] for the $\mathrm{SU}(2)_k$ Heisenberg model and generalized by the author to 'nested Bethe ansatz' systems [25]. Our results, through order $\mathrm{O}(1/L)$, are $\delta_j=[(2p-1)\ln(2)/4L\pi]\cosh[2\pi\alpha^p_j/(2p-1)]$. This last expression is in accord with the exact results of table 1 within 7% on average. The difference is due to the next order in $\mathrm{O}(1/L)$, not accounted for in this δ_j expression. We believe that the direct VBS extrapolations of the imaginary parts of the λ^p_j's roots and this last δ_j calculation support the stability of our FSH.

Now we would like to consider the finite-size corrections for the eigenspectra of these B_p NSLES models. The conformal anomaly c_p and the conformal dimensions X^p_n can be estimated from the finite-size corrections of the ground state energy $E^p_0(L)$ and of the excited states $E^p_n(L)$. More precisely [1, 26, 27], extrapolating the following estimates

$$c_p(L)=\frac{6L^2}{\zeta\pi}\left(e^p_\infty-\frac{E^p_0}{L}\right) \tag{6}$$

$$X^p_n(L)=\frac{L}{2\zeta\pi}(E^p_n(L)-E^p_0(L)) \tag{7}$$

we are able to compute c_p and X^p_n.

The ground state pertains to the sector where $m_1=m_2=\ldots=m_p=L$. Solving numerically the Bethe ansatz equations and substituting the λ^1_j solution into (2), we can compute the estimate (6). The results are shown in table 2 for $p=2,3$. These results suggest that the central charge c is given by $c_p=p+\frac{1}{2}$. We now concentrate our attention upon the spinless excited states for the B_2 NSLES model. The spinless excited states are parametrized by a set of integer numbers $(n_1,n_2,\ldots,n_p)$ that are defined as the difference between the number of particles in the ground state and in the respective

Table 2. Extrapolated and conjectured results for the central charge with $p = 2, 3$.

L	$p=2$	$p=3$
8	2.561 5960	3.529 9033
16	2.521 298	3.515 6908
24	2.512 5147	3.510 4361
32	2.508 9669	3.507 9645
40	2.507 0975	3.506 5418
Extr.	2.500 5 (7)	3.500 6 (5)
Conj.	2.5	3.5

excited state; i.e., $n_l = L - m_l$, $l = 1, 2, \ldots, p$. It is important to stress that the next-order corrections to (6), (7) are $O(1/\ln^3(L))$ and $O(1/\ln(L))$, respectively [2, 11, 28]. For the central charge, the respective logarithmic correction is treatable for a lattice size of about $L \sim 30$, but the correction $O(1/\ln(L))$ for excited states can give non-conclusive results for small lattice sizes. In order to avoid this problem, we first solve (1) using the FSH. In this case, using a standard technique [5, 6], the BAE can be manipulated analytically and we conclude that the central charge is $c_2 = 2$ and the conformal dimensions of the spinless excitations are $X_{\mathrm{FSH}}(n_1, n_2) = \frac{1}{4}(2n_1^2 + n_2^2 - 2n_1 n_2)$. Of course, the last results are not correct, since the FSH solution is valid only in the thermodynamic limit ($L \to \infty$) and we are here interested in $O(1/L^2)$ corrections. In order to get the correct estimate to the conformal dimensions $X(n_1, n_2)$ and at the same time avoid these logarithmic corrections, we solve the BAE for the associated excited state (n_1, n_2) and subtract the final solution of the respective FSH solution. In table 3 we show results for the difference $\delta X(n_1, n_2) = X(n_1, n_2) - X_{\mathrm{FSH}}(n_1, n_2)$ for $(n_1, n_2) = (1, 2); (2, 2)$. From this result and the use of the $X_{\mathrm{FSH}}(n_1, n_2)$ formula, one concludes that $X(1, 2) = 0.6247\,(5)$, $X(2, 2) = 1.0000\,(5)$. To test the consistency of this method, we solved the associated BAE for $L \sim 100$. Our results using direct extrapolations of (7), taking into account the respective logarithmic corrections, differ about 1% from this 'difference' method. We can interpret our results for the central charge and the conformal dimensions of these B_2 NSLES models as a composition of two bosonic fields ($c = 2$) and a Ising model (I) ($c = \frac{1}{2}$). The dimensions $X(1, 2)$ and $X(2, 2)$ can be rewritten as $X(1, 2) = X_{\mathrm{FSH}}(1, 2) \oplus (\frac{1}{16}, \frac{1}{16})_I$; $X(2, 2) = X_{\mathrm{FSH}}(2, 2) \oplus (0, 0)_I$. On the other hand, $c = 2.5$ and $x = \frac{5}{8}$ are the central charge and the lowest dimension, respectively, in the B_2 WZWN conformal field theory. Our results together with the previous results for the WZWN model [29] strongly suggest that the B_p NSLESs partition function is composed

Table 3. Extrapolated and conjectured results for the conformal dimensions $\delta X(1, 2)$ and $\delta X(2, 2)$.

L	$\delta X(1, 2)$	$\delta X(2, 2)$
8	0.119 0839	$5.653\,5781 - 10^{-2}$
16	0.122 7089	$1.972\,5120 - 10^{-3}$
20	0.123 2078	$1.447\,8173 - 10^{-3}$
24	0.123 5045	$1.140\,7027 - 10^{-3}$
28	0.123 6998	$9.427\,0641 - 10^{-4}$
Extr.	0.124 7 (5)	$4.65\,(2) - 10^{-5}$
Conj.	0.125	0

of $2p+1$ Ising models. In fact, we have observed that the BAE (1) are unstable for spinless excitations in the sector ($n_1=1$, $n_2=1$), in accordance with $x=\frac{5}{8}$ as the lowest dimension.

In conclusion, we have introduced the FSH solution for the B_p NSLES models. This idea has made it possible to determine exactly some properties in the thermodynamic limit, as well as finite-size corrections. We believe that this 'amusing' FSH clarifies the difference between non-simple laced and simple laced exactly solved models. We expect the FSH solution to be also important in calculating scattering properties and in finite temperature calculations [30].

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